ISRAEL JOURNAL OF MATHEMATICS, Vol. 68, No. 3, 1989

RIGID SETS IN THE PLANE

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ABSTRACT

A compact set in the plane is rigid with respect to a norm if the norm isometries of the set act transitively on it. We show that if a norm has an infinite rigid set, then, up to linear transformation, the norm is Euclidean and the set is a circle. Our methods also yield a new characterisation of the ellipse.

Introduction

Let E be a finite dimensional Banach space. A compact subset Γ of E is called *rigid* if it has the following property: given $a, b \in \Gamma$ there exists an isometry T of Γ onto itself with $Ta = b$. Clearly, if Γ is rigid with respect to the norm $\| \cdot \|$, and $L : E \to E$ is linear, then $L(\Gamma)$ is rigid with respect to the norm $\| L^{-1}(\cdot) \|$. For example, if E is a Euclidean space then a sphere is an infinite rigid set. The point of this paper is that, up to linear transformation, this is the only example of an infinite rigid set in a 2-dimensional Banach space.

Specifically, we prove

Main Theorem. *Let $(E, \| \cdot \|)$ be a 2-dimensional Banach space containing an infinite rigid set; then, up to linear transformation, the norm is Euclidean, and the rigid set is a circle.*

It has been shown by M. Akcoglu and U. Krengel ([1]) that a rigid set in $l^1(n)$ is finite $\forall n \geqq 1$; and by R. Sine ([4]) that a rigid set in a finite-dimensional Banach space whose unit sphere is polyhedral is finite.

Received April 26, 1988

The proof of the main theorem is in stages. First we prove that the infinite rigid set is actually a sphere of the norm, and that the norm is itself smooth (here, and throughout, smooth means C^1) and strictly convex. Then by exploiting rigidity, and an affine geometric property of rigid unit spheres, we prove that the norm must be Euclidean. A characterisation of ellipses is obtained as a corollary of our arguments.

Finally, the problem in higher dimensional Banach spaces is considered. We would like to thank H. Furstenberg, S. Hart, R. Livne and V. Milman for some helpful discussions, and Göttingen University for hospitality when this paper was begun.

§1. Γ is a topological circle

We shall denote $E = \mathbf{R}^2 = \mathbf{C}$ with a norm $\|\cdot\|$ sometimes denoted $N(\cdot)$; $\Gamma \subset E$ is an infinite rigid set. When we refer to distances, balls, spheres, isometries, arc-length, etc., we shall consider them defined (unless stated otherwise) with respect to the metric given by $\|\cdot\|$. We also have the usual norm (absolute value) $|\cdot|$ on C. We refer to it as the Euclidean norm and speak about Euclidean distances, Euclidean circles, etc.

We denote by G the group of all isometries of Γ onto itself. Equipped with the topology of uniform convergence, G is a compact topological group.

LEMMA 1.1. *There exists a compact convex set $B \subset E$ with a non-empty interior such that $\Gamma \subset \partial B$.*

PROOF. Let $\operatorname{diam}(\Gamma) = \sup\{\|a-b\| : a, b \in \Gamma\}$. Since Γ is compact, there exist $a, b \in \Gamma$ such that $\|a-b\| = \operatorname{diam}(\Gamma)$. For every $c \in \Gamma$ we can take $T \in G$ such that $T(a) = c$. Then $\|c - T(b)\| = \operatorname{diam}(\Gamma)$, $T(b) \in \Gamma$, and hence the ball B_c with centre $T(b)$ and radius $\operatorname{diam}(\Gamma)$ contains Γ and $c \in \partial B_c$. Therefore $\tilde{B} = \bigcap_{c \in \Gamma} B_c$ is a compact convex set for which $\Gamma \subset \partial \tilde{B}$. If $\operatorname{int} \tilde{B} \neq \varnothing$ then we take $B = \tilde{B}$. If not then $\tilde{B}$ is contained in a straight line. Since $\tilde{B}$ is bounded, we can find a compact convex set B with non-empty interior such that $\tilde{B} \subset \partial B$. □

The set ∂B is homeomorphic to the unit Euclidean circle $S^1 = \{z \in C : |z| = 1\}$.

Indeed, we may assume that $\mathbf{0} \in B$ and use the angular (i.e. polar) parametrisation of ∂B about $\mathbf{0}$:

$$\partial B = \{ p(t) = \rho(t)e^{it} : 0 \leqq t < 2\pi \}.$$

Here, $\rho : [0, 2\pi] \to \mathbf{R}_+$ is continuous, bounded above, and below, and may be extended to a bounded continuous function on $\mathbf{R}$ with period 2π.

Since B is convex, ∂B has one sided tangents at every point on ∂B. For $a \in \partial B$ we denote by S_a^+ the positively oriented, positive tangent to ∂B at a, and by S_a^-, the positively oriented negative tangent at a, and both by S_a if they coincide. The angle made by $S_{p(t)}^+$ with the x-axis is denoted by $\gamma(t)$, and $\gamma : [0, 2\pi) \to \mathbf{R}$ can be extended to a non-decreasing function $\gamma : \mathbf{R} \to \mathbf{R}$ satisfying $t < \gamma(t) < t + \pi$, and $\gamma(t + 2\pi) = \gamma(t) + 2\pi$. It follows that $\lim_{t \to s^+} \gamma(t) = \gamma(s)$, and that the angle made by $S_{p(t)}^-$ with the x-axis is, modulo 2π, $\gamma(t^-) \overset{\text{def}}{=} \lim_{s \to t^-} \gamma(s)$.

We also have that for $s \leqq t \leqq s + 2\pi$

$$\log \frac{\rho(t)}{\rho(s)} = \int_s^t \cot(\gamma(u) - u)du$$

whence

(a) $\int_0^{2\pi} \cot(\gamma(u) - u)du = 0$ as $\rho(t + 2\pi) = \rho(t)$,

(b) $\rho(t)$ has one-sided derivatives at every t,

(c) ∂B is C^1 if and only if $\gamma(t)$ is continuous.

We note that convex sets may be specified by specifying γ with the above properties.

LEMMA 1.2. *There exists a constant $\delta > 0$ such that if $\lambda, \mu \in \mathbf{R}$ are such that*

(1) *$\lambda < \mu$ and $\gamma(\mu^-) - \gamma(\lambda) < \delta$,*

then for each $v \in (\lambda, \mu)$ we have

(2) $\| p(\lambda) - p(v) \| < \| p(\lambda) - p(\mu) \|$

and

(3) $\| p(\mu) - p(v) \| < \| p(\lambda) - p(\mu) \|$.

PROOF. Set $s = \frac{1}{2}\inf\{|b| : \| b \| = 1\}$ and $s_0 = s/\sup\{|b| : \| b \| = 1\}$. Since $0 < s_0 \leqq \frac{1}{2}$, there exists $\delta \in (0, \pi/6)$ such that $\sin(\delta) = s_0$.

Assume that (1) holds and take $v \in (\lambda, \mu)$. We shall prove (2); the proof of (3) is analogous. Set $a = p(\lambda)$, $b = p(\mu)$, $r = \| a - b \|$. Let K and L be the sphere and the open ball respectively, both with the centre a and radius r; let M be the Euclidean circle with centre a and radius rs (see Fig. 1).

By the convexity of B and since $\gamma(\mu -) - \gamma(\lambda +) < \pi$, we have $p(v) \in \Delta abc$, where c is the point of intersection of S_a^+ with S_b^- (if $\gamma(\mu -) = \gamma(\lambda +)$ then

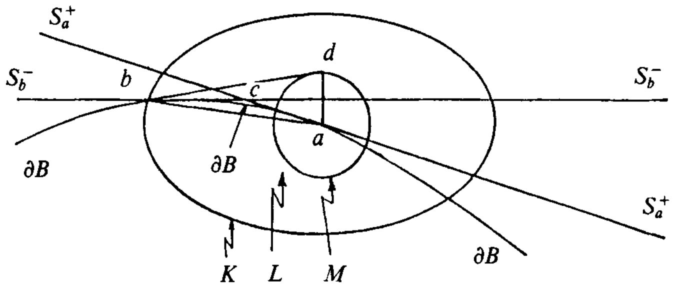

Fig. 1.

these lines coincide and we take any c between a and b and consider a degenerated triangle). Let $d \in M$ be such a point that bd is tangent to M and d and c lie on the same side of ab. The angles between the line ab and the lines S_a^+, S_b^- respectively, are at most $(\gamma(\mu-)-\gamma(\lambda+))<\delta$. On the other hand, $\sin \angle abd = rs/|b-a| \geqq s_0$ and $\cos \angle bad = rs/|b-a| \leqq \frac{1}{2}$. Hence, $\angle abd \geqq \delta$ and $\angle bad \geqq \pi/3 > \delta$. Consequently $c \in \Delta abd$ and therefore $\Delta abc \subset \Delta abd$. Since $a, d \in L$ and $b \in K$, we have $\Delta abd \subset L \cup \{b\}$. Hence we get $p(v) \in L$, which is equivalent to (2). □

LEMMA 1.3. $\Gamma = \partial B$.

PROOF. If a point $a \in \Gamma$ is isolated in Γ, then $T(a)$ is isolated in Γ for every $T \in G$ and hence all points of Γ are isolated in Γ. This contradicts our assumptions that Γ is compact and infinite. Therefore Γ has no isolated points.

Suppose that $\Gamma \neq \partial B$. Then there exists $x \in p^{-1}(\Gamma)$ and $\varepsilon_1 > 0$ such that $(x-\varepsilon_1, x) \cap p^{-1}(\Gamma) = \varnothing$. Let δ be as in Lemma 1.2. There exists $\varepsilon_2 > 0$ such that $\gamma(x+\varepsilon_2-) - \gamma(x+) < \delta$. Finally, there exists $r_0 > 0$ such that the intersection of the ball having centre $p(x)$ and radius r_0 with ∂B is contained in $p((x-\varepsilon_1, x+\varepsilon_2))$.

Since $p(x)$ is not isolated in Γ, there exist $y, z \in p^{-1}(\Gamma)$ such that $x < z < y$ and $\| p(x) - p(y) \| < r_0$, $\| p(x) - p(z) \| < r_0$ (see Fig. 2).

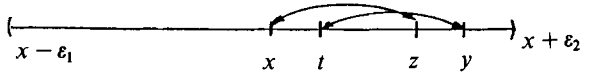

Fig. 2.

There exists $T \in G$ such that $T(p(z)) = p(x)$.

If $u \in p^{-1}(\Gamma)$ is such that $\| p(u) - p(z) \| < r_0$ then there exists $v \in (x - \varepsilon_1, x + \varepsilon_2)$ such that $T(p(u)) = p(v)$. Since $\| T(p(u)) - p(x) \| < r_0$, we get $v \in [x, x + \varepsilon_2)$.

Applying this to $u = x$ we get $v \in [x, x + \varepsilon_2)$ such that

$$\| p(v) - p(x) \| = \| T(p(x)) - T(p(z)) \| = \| p(x) - p(z) \| .$$

By Lemma 1.2, we obtain $v = z$. Hence $T(p(x)) = p(z)$.

When we take $u = y$ instead, we get some point $t(= v)$ with $t \in [x, x + \varepsilon_2)$ and $T(p(y)) = p(t)$. Since by Lemma 1.2, $\| p(z) - p(y) \| < \| p(x) - p(y) \|$, we get $\| p(x) - p(t) \| < \| p(x) - p(y) \|$ and again, by Lemma 1.2, we obtain $t < y$. Now by using Lemma 1.2 twice, we get $\| p(t) - p(z) \| < \| p(x) - p(y) \|$. On the other hand,

$$\| p(x) - p(y) \| = \| T(p(x)) - T(p(y)) \| = \| p(z) - p(t) \| ,$$

a contradiction. Hence $\Gamma = \partial B$. □

Let H be the subgroup of G consisting of all orientation preserving isometries of Γ. Then clearly H is a clopen normal subgroup of index 2.

Arc length distance is defined on Γ by

$$\alpha(s, t) = \tilde{\alpha}(p(s), p(t)) = \int_s^t \| p'(u) \| \, du = \lim_{\lambda \to 0} \sum_{k=0}^{n-1} \| p(u_{k+1}) - p(u_k) \|$$

where $n \geqq 1$, $s = u_0 < u_1 < \cdots < u_n = t < s + 2\pi$ and $\lambda = \max_k (u_{k+1} - u_k)$.

Clearly any (norm) isometry of Γ preserves arc-length whence G is abelian.

Also, for every $a, b \in \Gamma$ there is a unique orientation preserving homeomorphism T such that $Ta = b$, which preserves arc-length.

Lemma 1.4. *For every $a, b \in \Gamma$ there is a unique $T \in H$ such that $Ta = b$.*

Proof. Fix $a \in \Gamma$ and let $j : H \to \Gamma$ be defined by $jT = Ta$ $(T \in H)$. Then j is a continuous map, whose restriction to H is 1–1. Let $\Gamma_0 = j(H)$. We must prove that $\Gamma_0 = \Gamma$. This is clear in case $H = G$. Otherwise $\exists\, T_0 \in G \setminus H$, whence $G = H \cup T_0 H$ and $\Gamma = \Gamma_0 \cup T_0 \Gamma_0$. Since Γ_0 is closed, it follows that int $\Gamma_0 \neq \varnothing$. Now, Γ_0 is easily seen to be open because $j(gK) = gj(K)$ for $g \in G$, $K \subseteq G$. By connectedness $\Gamma_0 = \Gamma$. □

Lemma 1.5. (a) *An orientation preserving homeomorphism of Γ is an isometry if and only if it preserves arc-length.*

(b) *H is isomorphic, as a topological group to the circle group* $\mathbf{R}/\mathbf{Z}$.

PROOF. (a) As remarked above an orientation preserving isometry of Γ preserves arc-length. Conversely if T is orientation preserving and preserves arc-length, then the unique element S of H which maps a onto Ta (for some $a \in \Gamma$) is also an orientation preserving arc-length preserving transformation (mapping a onto Ta), and, being unique in this role, therefore coincides with T.

(b) This is an immediate corollary of (a). □

We may parametrise Γ by arc-length. If the length of Γ is $2L$, then there is a bijection $\psi: [0, 2L) \to \Gamma$ such that the arc length distance from $\psi(s)$ to $\psi(s+t)$ $(s \leqq t \leqq s + 2L)$ is t. Here $\psi(t) = p(\phi^{-1}(t))$ where $\phi(t) = \int_0^t \| p'(u) \| \, du = \alpha(0, t)$. We shall sometimes denote $\phi^{-1} = \varphi$. The map ψ may be extended periodically to $\psi: \mathbf{R} \to \Gamma$.

We write $H = \{T_s : s \in \mathbf{R}\}$ where $T_s \psi(t) = \psi(t + s)$.

LEMMA 1.6. *A convex closed curve Γ is rigid if and only if $\| \psi(s+t) - \psi(s) \|$ is independent of s for every t where ψ is the arc-length parametrisation of Γ.*

PROOF. If Γ is rigid then by Lemma 1.5(a) $\psi(s) \mapsto \psi(s+t)$ is a norm isometry, whence $\| \psi(s+t) - \psi(s) \|$ does not depend on s; and this statement, for every t, in turn implies that the maps $\psi(s) \mapsto \psi(s+t)$ are norm isometries, whence Γ is rigid. □

§2. The norm is smooth, and the rigid set is a sphere

LEMMA 2.1. *Γ is a sphere in the metric given by* $\| \cdot \|$.

PROOF. There is $x_0 \in (0, 2L)$ such that $\| \psi(x_0) - \psi(0) \| = \mathrm{diam}(\Gamma)$.

Let $x \in \mathbf{R}$ be such that γ is continuous at both $\varphi(x)$ and $\varphi(x + x_0)$. Set $a = \psi(x)$, $b = \psi(x + x_0)$. Then $\| a - b \| = \mathrm{diam}(\Gamma)$ and S_a, S_b are tangent to Γ at a and b respectively.

Let v be a non-zero vector parallel to S_a. For small $t \in \mathbf{R}$ the line L_t parallel to ab and passing through $a + tv$ intersects the lines S_a and S_b at points $a_1(t) = a + tv$ and $b_1(t)$ respectively and Γ at points $a_2(t)$ (close to a) and $b_2(t)$ (close to b) (see Fig. 3). Since S_a and S_b are tangent to Γ, we get

$$\lim_{t \to 0} \frac{\| a_1(t) - a_2(t) \|}{|t|} = \lim_{t \to 0} \frac{\| b_1(t) - b_2(t) \|}{|t|} = 0.$$

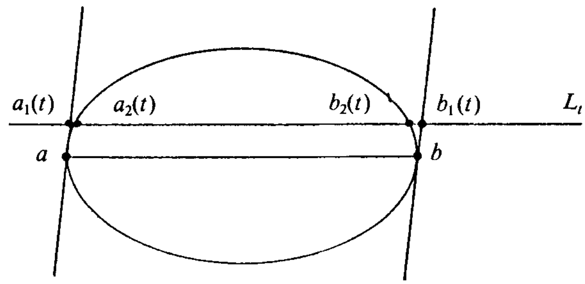

Fig. 3.

Moreover,

$$\| a_1(t) - b_1(t) \| = \| a_1(t) - a_2(t) \| + \| a_2(t) - b_2(t) \| + \| b_1(t) - b_2(t) \| .$$

Since $\| a_2(t) - b_2(t) \| \leqq \operatorname{diam}(\Gamma) = \| a - b \|$, we get

$$\limsup_{t \to 0} \frac{\| a_1(t) - b_1(t) \| - \| a - b \|}{|t|} \leqq 0.$$

This implies that S_b is parallel to S_a.

For small $t \in \mathbf{R}$ draw the lines M_1 and M_2 parallel to ab through points $\psi(x + t)$ and $\psi(x + x_0 + t)$ respectively. M_1 interesects S_a at a point $a + s(t)v$ and M_2 intersects S_b at a point $b + r(t)v$ for some $s(t), r(t) \in \mathbf{R}$ (see Fig. 4).

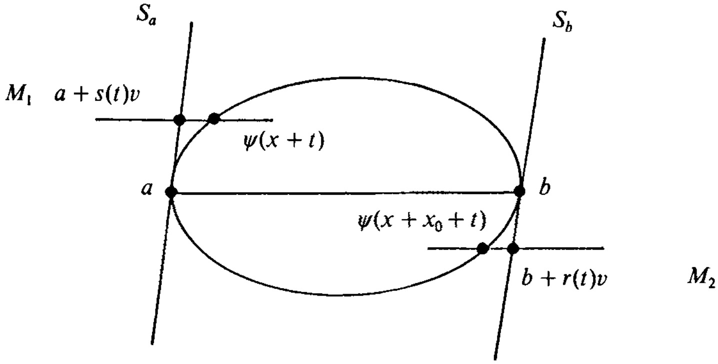

Fig. 4.

Since ψ is the arc-length parametrisation of Γ, and S_a and S_b are tangent to Γ and parallel to each other, we obtain

$$\lim_{t\to 0}\frac{\|\psi(x+t)-a\|}{|t|}=1=\lim_{t\to 0}\frac{\|\psi(x+x_0+t)-b\|}{|t|}$$

and

$$\lim_{t\to 0}\frac{\|\psi(x+t)-(a+s(t)v)\|}{|t|}=\lim_{t\to 0}\frac{\|\psi(x+x_0+t)-(b+r(t)v)\|}{|t|}=0.$$

It follows that $\lim_{t\to 0}|s(t)/t| = 1/\|v\| = \lim_{t\to 0}|r(t)/t|$. Clearly, $s(t)$ and $r(t)$ have opposite signs. Therefore

$$\lim_{t\to 0}\frac{s(t)+r(t)}{|t|}=0.$$

We have

$$\begin{aligned}&\left\|\frac{\psi(x+t)+\psi(x+x_0+t)}{2}-\frac{a+b}{2}\right\|\\ &\quad\leqq \tfrac{1}{2}\|\psi(x+t)-(a+s(t)v)\| + \tfrac{1}{2}\|\psi(x+x_0+t)-(b+r(t)v)\|\\ &\qquad+\tfrac{1}{2}(s(t)+r(t))\|v\|.\end{aligned}$$

Hence,

$$\lim_{t\to 0}\frac{1}{|t|}\left\|\frac{\psi(x+t)+\psi(x+x_0+t)}{2}-\frac{a+b}{2}\right\|=0.$$

Thus the function $F:\mathbf{R}\to E$, given by

$$F(x)=\frac{\psi(x)+\psi(x+x_0)}{2},$$

has derivative 0 at all points x such that γ is continuous at both $\varphi(x)$ and $\varphi(x+x_0)$, i.e. at all points except a countable number of them. The function ψ is Lipschitz continuous, and hence F is also Lipschitz continuous. Consequently, F is constant. Since $\|\psi(x)-F(x)\| = \frac{1}{2}\operatorname{diam}(\Gamma)$ for all $x\in\mathbf{R}$, Γ is the sphere with centre $F(0)$ and radius $\frac{1}{2}\operatorname{diam}(\Gamma)$. □

Without loss of generality, we may assume that Γ is the unit sphere, and B is the unit ball around zero. One of the important consequences of this is that

Γ is symmetric with respect to 0. Hence, $p(t+\pi) = -p(t)$, and $\gamma(t+\pi) = \gamma(t) + \pi$. Also, for each $a \in \Gamma$ both arcs between a and $-a$ have the same length. Hence $\psi(t+L) = -\psi(t)$.

LEMMA 2.2. *The norm (i.e. its unit ball) is strictly convex and smooth.*

PROOF. Assume that $a, b \in \Gamma$ are such that $\| a - b \| = 2$. Draw the line parallel to ab and passing through 0. It intersects Γ at two points, c and d. By symmetry of Γ, $d = -c$, so if $c = \psi(x)$ then $d = \psi(x+L)$. Since $\gamma(x+\pi) = \gamma(x) + \pi$, S_c and S_d are parallel (see Fig. 5).

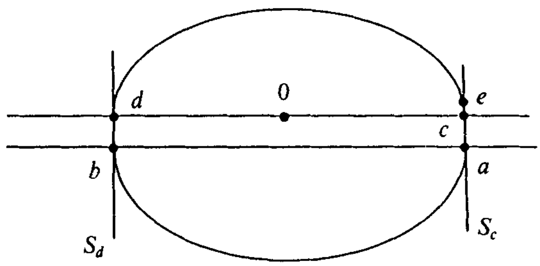

Fig. 5.

Since also $\| c - d \| = 2$ and Γ lies between S_c and S_d, we have either $a \in S_c$, $b \in S_d$ or $a \in S_d$, $b \in S_c$.

Suppose that $\{a, b\} \neq \{c, d\}$. Without loss of generality we may assume that $a \in S_c$, $b \in S_d$. By symmetry, $e = -b \in S_c$. Then c lies on the segment ae and therefore it is not an extremal point of B.

If we replace a and b by $T_t(a)$ and $T_t(b)$ respectively, then the point $c(t)$ which replaces c will depend continuously on t. Moreover, as t varies from 0 to $2L$, $c(t)$ passes through the whole of Γ. If $a = \psi(y)$ then, since $b \neq -a$, we have $b \neq \psi(y+L)$. Therefore $T_t(a) = \psi(y+t)$, but $T_t(b) \neq \psi(y+t+L)$. Thus, $\{T_t(a), T_t(b)\} \neq \{c(t), -c(t)\}$, and hence $c(t)$ is not an extremal point of B. Therefore no point of ∂B is extremal, a contradiction.

Thus if $\| a - b \| = 2$ for some $a, b \in \Gamma$ then $b = -a$. However, if B is not strictly convex then there exists $a \in \Gamma$ which is not an extremal point of B. Then any straight line parallel to $a\mathbf{0}$ and sufficiently close to it intersects Γ at two points which are at distance 2 from each other. This is a contradiction, since these points are not symmetric with respect to $\mathbf{0}$. Therefore B is strictly convex.

We now prove smoothness, Recall that $\psi(s) = p(\varphi(s))$ where $\int_0^{\varphi(s)} \| p'(u) \| \, du = s$. Clearly

$$\frac{1}{h} \| (\psi(t+h) - \psi(t)) \| \to 1 \text{ and } \frac{1}{h} \| (\psi(t-h) - \psi(t)) \| \to 1 \quad \text{as } h \to 0^+.$$

The directions of $h^{-1}(\psi(t+h) - \psi(t))$ and $h^{-1}(\psi(t) - h))$ converge to $e^{i\gamma(\varphi(t)^+)}$ and $e^{i\gamma(\varphi(t)^-)}$ respectively. Thus, $\psi(s)$ has one-sided derivatives $\psi^+(s)$ and $\psi^-(s)$ everywhere, which coincide at s if and only if γ is continuous at $\varphi(s)$, and moreover $\| \psi^+ \| = \| \psi^- \| = 1$.

Also, for every s,

$$\frac{1}{h} \| \psi(s+h) - \psi(s-h) \| \to \| \psi^+(s) + \psi^-(s) \| \qquad \text{as } h \to 0.$$

By strict convexity and $\| \psi^\pm \| = 1$, this limit is equal to 2 if and only if $\psi^+(s) = \psi^-(s)$, i.e. γ is continuous at $\varphi(s)$.

There are points s such that γ is continuous at $\varphi(s)$, and hence for which this limit is equal to 2. By Lemma 1.6, this limit is 2 for every s, whence γ is continuous and Γ is smooth. □

§3. Rigidity and the M' property

In the next three sections, we prove

THEOREM 3.1. *If N is a strictly convex, smooth norm whose unit sphere is rigid, then there is a linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ such that $N \circ L(x, y) = \sqrt{x^2 + y^2}$.*

To do this, we shall need a strengthening of rigidity, and a geometric property of a rigid unit sphere which we develop in this section.

Let Γ be a smooth (C^1), strictly convex closed curve in $\mathbf{R}^2$, and let O be a point in its interior. Given a point A on Γ, we construct a new point A' on Γ such that the directed line $\overrightarrow{OA'}$ has the same direction as the tangent to Γ at A in the positive direction. It follows from smoothness and strict convexity of Γ that the map $A \mapsto A'$ is a homeomorphism of Γ. Given two points B, C on Γ, there is a unique point $M(B, C)$ on Γ with the property that the positive tangent to Γ at $M(B, C)$ has the same direction as the directed line $\overrightarrow{BC}$.

We say that the curve Γ *has the M' property about O* if for every A, B on Γ

$$M(A', B') = M(A, B)'.$$

It is not hard to see that if Γ has the M' property about O, and L is an affine map of the plane, then $L(\Gamma)$ has the M' property about $L(O)$, and indeed

$$L(A)' = L(A'), \quad M(L(B), L(C)) = L(M(B, C)).$$

Clearly, circles have the M' property about their centres, whence ellipses have the M' property about their points of symmetry (being affine images of circles). We shall prove later that if a curve has the M' property about some point, then the curve is an ellipse, and the point is its point of symmetry. Our first task here is to prove

LEMMA 3.1. *Let N be a smooth strictly convex norm on $\mathbf{R}^2$ with unit sphere V. Then V is rigid if and only if V has the M' property about its centre.*

PROOF. Let $p(t) = \rho(t)e^{it}$ be the angle parametrisation of V about its centre $\mathbf{0}$. For $s \leqq t \leqq s + 2\pi$, write $M(p(s), p(t)) = p(\theta(s, t))$ and $(p(s))' = p(\gamma(s))$. Then $\gamma(s) = s + \cot^{-1} a'(s)$ where $a = \log \rho$, and the M' property about $\mathbf{0}$ is

$$\gamma\theta(s, t) = \theta(\gamma(s), \gamma(t)).$$

Let $2L$ be the (norm) arc length of Γ, i.e. $2L = \int_0^{2\pi} N(p'(t))dt$.

Next, let $\varphi(s)$ be that increasing diffeomorphism of $[0, 2L]$ onto $[0, 2\pi]$ such that the arclength parametrisation $\psi(s) = p(\varphi(s))$; then $N(\psi'(s)) = 1$ and hence $\psi'(s) = p(\gamma\varphi(s))$.

Now by Lemma 1.6, V is rigid if and only if $N(\psi(s + t) - \psi(s))$ does not depend on s, i.e.

$$\frac{d}{ds} N(\psi(s + t) - \psi(s)) \equiv 0.$$

Now

$$\frac{d}{ds} N(\psi(s + t) - \psi(s)) = \nabla N(\psi(s + t) - \psi(s)) \cdot (\psi'(s + t) - \psi'(s))$$

$$= \nabla N(p(\varphi(s + t)) - p(\varphi(s)) \cdot (p(\gamma\varphi(s + t)) - p(\gamma\varphi(s)).$$

Since $N(re^{i\theta}) = rN(e^{i\theta})$ $(r > 0)$ we have that

$$\nabla N(re^{i\theta}) = \nabla N(e^{i\theta}) = \nabla N(p(\theta))$$

and this is the outward normal to V at $p(\theta)$.

Thus $\nabla N(p(\varphi(s+t))-p(\varphi(s)))$ is the outward normal to V at $p(\gamma\theta(\varphi(s), \varphi(s+t)))$, and

$$\frac{d}{ds}N(\psi(s+t)-\psi(s))\equiv 0$$

if and only if the direction of $p(\gamma\varphi(s+t))-p(\gamma\varphi(s))$ (which is $\gamma\theta(\gamma\varphi(s), \gamma\varphi(s+t))$) is perpendicular to that of the outward normal to V at $p(\gamma\theta(\varphi(s), \varphi(s+t)))$, or parallel to the tangent to V at this point. This means $\gamma^2\theta(\varphi(s), \varphi(s+t))=\gamma\theta(\gamma\varphi(s), \gamma\varphi(s+t))$ which by monotonicity of γ is the same as $\gamma\theta(\varphi(s), \varphi(s+t))=\theta(\gamma\varphi(s), \gamma\varphi(s+t))$; i.e. the M' property for V about $\mathbf{0}$. □

LEMMA 3.2. *Suppose that V is smooth, strictly convex, symmetric about* $\mathbf{0}$ *and has the M' property about* $\mathbf{0}$, *then* $\gamma^2(t)=t+\pi$ $\forall t$, *where* $\gamma(t)=t+\cot^{-1}a'(t)$, $a=\log\rho$, $p(t)=\rho(t)e^{it}$ *is the angle parametrisation of V about* $\mathbf{0}$.

PROOF. We begin with a claim. (This is a lemma of Auerbach, see [2], p. 68.)

CLAIM. If $\Gamma=\{p(t)=\rho(t)e^{it}: 0\leqq t\leqq 2\pi\}$ is smooth and convex then $\exists t\in[0,2\pi]$ such that $\gamma^2(t)=t+\pi$. That is, $\exists a\in\Gamma$ with $a''=-a$.

PROOF. Let $a, b\in\Gamma$ be such that the area of $\Delta\mathbf{0}ab$ is maximal. Draw the straight line through b parallel to $\mathbf{0}a$. If $c\in\Gamma$ then, by assumption, the area of $\Delta\mathbf{0}ac$ is less than or equal to the area of $\Delta\mathbf{0}ab$, whence the perpendicular distance of c to the line $\mathbf{0}a$ is less than or equal to that of b to $\mathbf{0}a$. This shows that the line parallel to $\mathbf{0}a$ through $\mathbf{b}$ is tangent to Γ. Similarly, the tangent to Γ at a is parallel to $\mathbf{0}b$. Without loss of generality, $a=p(s)$, $b=p(t)$ where $s<t<s+\pi$, and we obtain $t=\gamma(s)$, $\gamma(t)=s+\pi$ whence $\gamma^2(s)=s+\pi$. □

To complete the proof of the lemma, let $K=\{s\in\mathbf{R}:\gamma^2(s)=s+\pi\}$. Then K is symmetric closed, γ-invariant and non-empty. If $K^c\neq\varnothing$ then $\exists s, t\in K$, $s<t<s+\pi$ such that $(s,t)\cap K=\varnothing$.

But $s<\theta(s,t)<t$ and

$$\begin{aligned}\gamma^2\theta(s,t)&=\theta(\gamma^2 s, \gamma^2 t) && \text{by property } M',\\ &=\theta(s+\pi, t+\pi) && \text{since } s,t\in K,\\ &=\theta(s,t)+\pi && \text{by symmetry,}\end{aligned}$$

contradicting $\theta(s, t) \notin K$, and proving the lemma. □

LEMMA 3.3. *Let N be a smooth, strictly convex norm on $\mathbf{R}^2$ whose unit sphere V is rigid.*

Let $\psi(s)$ $(0 \leqq s \leqq 2L)$ be the arc-length parametrisation of V such that $N(\psi'(s)) = 1$. Then for every $s, t \in [0, 2L]$

$$N(\psi(s+t) - \psi(s)) = Q(t),$$

where $Q : [0, 2L] \to \mathbf{R}_+$ is smooth, $Q'(t) > 0$ on $(0, L)$ and $Q(L+t) = Q(L-t)$ for $t \in [0, L]$.

PROOF. Since V is rigid, we have that

$$N(\psi(s+t) - \psi(s)) = Q(t) \qquad \text{where } Q : [0, 2L] \to \mathbf{R}_+$$

is smooth, whence $(\psi(s+2L) = \psi(s))$:

$$\begin{aligned} Q(L+t) &= N(\psi(s+t+L) - \psi(s)) \\ &= N(\psi(s+2L) - \psi(s+t+L)) \\ &= Q(L-t). \end{aligned}$$

If $p(s) = \rho(s)e^{is}$ is the angle parametrisation of V, then $\psi(s) = p(\varphi(s))$ where $\varphi : [0, 2L] \to [0, 2\pi]$ is an increasing diffeomorphism, and $\psi'(s) = p(\gamma\varphi(s))$.

By symmetry, $\varphi(s+L) = \varphi(s) + \pi$.

Since $\gamma(t) = t + \cot^{-1} a'(t)$ where $a = \log \rho$, there exists s_0 such that $\gamma\varphi(s_0) = \varphi(s_0) + \pi/2$ (take $\varphi(s_0)$ stationary for a) whence, by Lemma 3.2, $\gamma(\varphi(s_0) + \pi/2) = \varphi(s_0) + \pi = \varphi(s_0 + L)$. Now,

$$\begin{aligned} Q'(t) &= \frac{d}{dt} N(\psi(s_0 + t) - \psi(s_0)) \\ &= \nabla N(\psi(s_0 + t) - \psi(s_0)) \cdot \psi'(s_0 + t) \\ &= \nabla N(p(\varphi(s_0 + t)) - p(\varphi(s_0))) \cdot p(\gamma\varphi(s_0 + t)) \\ &= \nabla N(p(\varphi(s_0 + t)) - p(\varphi(s_0))) \cdot p(\gamma\varphi(s_0)), \end{aligned}$$

since

$$\frac{d}{ds} N(\psi(s+t) - \psi(s)) = 0.$$

Now the direction of $\nabla N(p(\varphi(s_0 + t)) - p(\varphi(s_0)))$ is the outward normal to

V at $p(\gamma\theta(\varphi(s_0), \varphi(s_0+t)))$ which is $\gamma^2\theta(\varphi(s_0), \varphi(s_0+t)) - \pi/2$ and is actually $\pi/2 + \theta(\varphi(s_0), \varphi(s_0+t))$ by Lemma 3.2.

Thus $Q'(t) > 0$ if and only if

$$\cos(\theta(\varphi(s_0), \varphi(s_0+t)) - \varphi(s_0)) > 0 \qquad (\text{since } \gamma\varphi(s_0) = \varphi(s_0) + \pi/2).$$

For $0 < t < L$, we have that

$$\begin{aligned} \varphi(s_0) &< \theta(\varphi(s_0), \varphi(s_0+t)) \\ &< \theta(\varphi(s_0), \varphi(s_0+L)) \\ &= \theta(\varphi(s_0), \varphi(s_0)+\pi) \\ &= \gamma^{-1}(\varphi(s_0)+\pi) = \varphi(s_0) + \pi/2, \end{aligned}$$

whence $0 < \theta(\varphi(s_0), \varphi(s_0+t)) - \varphi(s_0) < \pi/2$ and $Q'(t) > 0$. □

§4. Consequences of rigidity

In this section we give that part of the proof of Theorem 1 which exploits rigidity. The other part, which exploits property M', is performed in the next section.

Let N be the smooth, strictly convex norm and V its unit sphere with angle parametrisation $p(t) = \rho(t)e^{it}$. For $s \leqq t \leqq s + 2\pi$ we have the N-arc distance from $p(s)$ to $p(t)$ defined by

$$\alpha(s, t) = \int_s^t N(p'(u))du.$$

Note that $\alpha(s, s+2\pi) = \alpha(0, 2\pi) = 2L$ for every s and, by symmetry, and additivity of α,

$$\alpha(s, s+\pi) = L \qquad \text{for every } s.$$

By Lemma 3.3, rigidity of V is equivalent to the property that if $s \leqq t \leqq s + \pi$, $s' \leqq t' \leqq s' + \pi$ then

$$N(p(t) - p(s)) = N(p(t') - p(s')) \qquad \text{if and only if } \alpha(s, t) = \alpha(s', t').$$

Observe that if $\rho(s) = \rho(t)$, $s < t < s + \pi$ then

$$N(p(t) - p(s)) = \frac{2\rho(s)}{\rho\left(\frac{\pi}{2} + \frac{s+t}{2}\right)} \sin\left(\frac{t-s}{2}\right). \tag{4.1}$$

LEMMA 4.1. *Let $v = 1$ or 2, $m \geqq 2$ if $v = 1$, and $m \geqq 3$ if $v = 2$. If*

$$\rho\left(\frac{(k+2v)\pi}{2^m}\right) = \rho\left(\frac{k\pi}{2^m}\right) \qquad \forall k$$

then

$$\rho\left(\frac{(k+v)\pi}{2^m}\right) = \rho\left(\frac{k\pi}{2^m}\right) \qquad \forall k.$$

PROOF. Note that under our assumptions $v \mid 2^{m-2}$. Set $b_k = k\pi/2^m$, then $\rho(b_{k+2v}) = \rho(b_k)$ and by (4.1)

$$N(p(b_{k+2v}) - p(b_k)) = \frac{2\rho(b_k)\sin b_v}{\rho(b_{k+v})}. \tag{4.2}$$

Thus, $N(p(b_{k+2v}) - p(b_k)) = N(p(b_{k+4v}) - p(b_{k+2v}))$ and by rigidity $\alpha(b_k, b_{k+2v}) = \alpha(b_{k+2v}, b_{k+4v})$. For every k, by additivity of α,

$$L = \alpha(b_k, b_k + \pi) = \alpha(b_k, b_{k+2^m}) = \sum_{l=0}^{R-1} \alpha(b_{k+2lv}, b_{k+2(l+1)v})$$

$$= R\alpha(b_k, b_{k+2v})$$

where $R = 2^{m-1}/v \in \mathbb{N}$. Thus

$$\alpha(b_k, b_{k+2v}) = \frac{Lv}{2^{m-1}} \qquad \forall k.$$

Therefore, by rigidity,

$$N(p(b_{k+2v}) - p(b_k)) = N(p(b_{k+3v}) - p(b_{k+v}))$$

whence, by (4.2),

$$\frac{\rho(b_k)}{\rho(b_{k+v})} = \frac{\rho(b_{k+v})}{\rho(b_{k+2v})}.$$

But $\rho(b_{k+2v}) = \rho(b_k)$ and therefore $\rho(b_k)^2 = \rho(b_{k+v})^2$. □

LEMMA 4.2. *If $\rho(k\pi/2) = 1$ $\forall k$ and $\exists a > 0$ such that $\rho(\pi/4 + k\pi/2) = a$ $\forall k$ then $\rho(s) = 1$ $\forall s$ and $N(x, y) = \sqrt{x^2 + y^2}$.*

PROOF. By Lemma 4.1 with $v = 1$ and $m = 2$, $\rho(k\pi/4) = 1 = a$ $\forall k$. We prove by induction that for $n \geqq 2$,

$$\rho(k\pi/2^n) = 1 \qquad \forall k. \tag{4.3}$$

This is true for $n=2$. Suppose that (4.3) is true for some $n \geqq 2$ and let $c_k = k\pi/2^{n+1}$. We have that $\rho(c_{2k}) = 1$ $\forall k$, and so by (4.1)

$$N(p(c_{2k+4}) - p(c_{2k})) = 2\sin c_2 = N(p(c_{2k+6}) - p(c_{2k+2})) \qquad \forall k.$$

By rigidity, and additivity of α,

$$\begin{aligned}\alpha(c_{2k}, c_{2k+2}) &= \alpha(c_{2k}, c_{2k+6}) - \alpha(c_{2k+2}, c_{2k+6}) \\ &= \alpha(c_{2k}, c_{2k+6}) - \alpha(c_{2k}, c_{2k+4}) \\ &= \alpha(c_{2k+4}, c_{2k+6}) \qquad \forall k.\end{aligned}$$

Again, by rigidity,

$$N(p(c_{2k+2}) - p(c_{2k})) = N(p(c_{2k+6}) - p(c_{2k+4})) \qquad \forall k.$$

Using (4.1), we obtain

$$\rho(c_{2k+1}) = \rho(c_{2k+5}).$$

That is, $\rho(c_l) = \rho(c_{l+4})$ for l odd. Thus ($\rho(c_{2k}) = 1$), $\rho(c_k) = \rho(c_{k+4})$ $\forall k$.

By Lemma 4.1 with $m = n+1$ ($\geqq 3$) and $\nu = 2$,

$$\rho(c_k) = \rho(c_{k+2}) \qquad \forall k$$

and again by Lemma 4.1 with $m = n+1$ and $\nu = 1$

$$\rho(c_k) = \rho(c_{k+1}) \qquad \forall k.$$

By induction, (4.3) is true $\forall n \geqq 2$, whence by continuity of ρ, $\rho(s) = 1$ $\forall s$, and $N(x,y) = \sqrt{x^2+y^2}$. □

§5. Consequences of the M' property

In this section we complete the proof of Theorem 3.1 and hence of the main theorem, by showing that a smooth, strictly convex norm on $\mathbf{R}^2$ whose unit sphere is rigid has a linear translate which satisfies the conditions of Lemma 4.1. This is done using the M' property. We also show that the M' property characterises ellipses.

LEMMA 5.1. *Suppose that V is a smooth, strictly convex closed curve in $\mathbf{R}^2$, symmetric about $\mathbf{0}$ and having the M' property about $\mathbf{0}$, and let $p(t) = \rho(t)e^{it}$ be its angle parametrisation.*

Then there exists a linear transformation L such that if $p_L(t) = \rho_L(t)e^{it}$ is the *angle parametrisation of $L(V)$, then $\rho_L(k\pi/2) = 1\ \ \forall k$, and, for some $a > 0$, $\rho_L(k\pi/2 + \pi/4) = a\ \forall k$.*

PROOF. By rotation about $\mathbf{0}$ (a linear transformation), we may ensure that $\mathbf{0}$ is a stationary point for ρ, whence $\gamma(0) = \pi/2$, and by Lemma 3.2 $\gamma(\pi/2) = \gamma^2(0) = \pi$, whence $\gamma(k\pi/2) = k\pi/2 + \pi/2$.

By symmetry of V, $\rho(0) = \rho(\pi)$ and $\rho(\pi/2) = \rho(3\pi/2)$.

Applying the linear transformation $(x, y) \mapsto (x/\rho(0), y/\rho(\pi/2))$ we preserve the property $\gamma(k\pi/2) = k\pi/2 + \pi/2$ (because this transformation maps horizontal lines onto horizontal lines, and vertical lines onto vertical lines), and obtain, in addition, that

$$\rho(k\pi/2) = 1 \quad \forall k.$$

Next, $\gamma\theta(0, \pi/2)$ has the direction of the line $p(0)\,p(\pi/2)$, which is $3\pi/4$ since $\rho(0) = \rho(\pi/2)$; that is, $\gamma\theta(0, \pi/2) = 3\pi/4$.

Similarly, $\gamma\theta(k\pi/2, k\pi/2 + \pi/2) = k\pi/2 + 3\pi/4$ whence

$$\begin{aligned}
\theta\left(\frac{k\pi}{2}, \frac{k\pi}{2} + \frac{\pi}{2}\right) &= \gamma^2\theta\left(\frac{k\pi}{2}, \frac{k\pi}{2} + \frac{\pi}{2}\right) - \pi \qquad \text{by Lemma 3.2} \\
&= \gamma\theta\left(\frac{(k+1)\pi}{2}, \frac{(k+1)\pi}{2} + \frac{\pi}{2}\right) - \pi \qquad \text{by the } M' \text{ property} \\
&= \frac{(k+1)\pi}{2} + \frac{3\pi}{4} - \pi \qquad \text{by the above} \\
&= \frac{k\pi}{2} + \frac{\pi}{4}.
\end{aligned}$$

Thus

$$\gamma(k\pi/4) = k\pi/4 + \pi/2 \quad \forall k.$$

We have, by symmetry, that $\rho(\pi/4) = \rho(5\pi/4)$ and $\rho(3\pi/4) = \rho(7\pi/4)$ and we must show that

$$\rho(\pi/4) = \rho(3\pi/4).$$

If this were false, we could ensure, by a possible rotation by $\pi/2$, that $\rho(\pi/4) > \rho(3\pi/4)$. This would imply that the direction of the chord $p(\pi/4)\,p(3\pi/4)$ (which is $\gamma\theta(\pi/4, 3\pi/4)$) is larger than π, whence (as $\gamma(\pi/2) = \pi$) we would have

$$\theta(\pi/4, 3\pi/4) > \pi/2.$$

In a similar manner, we would obtain that

$$\theta(3\pi/4, 5\pi/4) < \pi.$$

This is impossible, as $\theta(\pi/4, 3\pi/4) > \pi/2$ implies that

$$\begin{aligned} \pi = \gamma(\pi/2) &< \gamma\theta(\pi/4, 3\pi/4) && \text{as } \gamma \text{ increases} \\ &= \theta(3\pi/4, 5\pi/4) && \text{by the } M' \text{ property} \\ &< \pi. \end{aligned}$$

Thus $\rho(\pi/4) = \rho(3\pi/4)$.

Theorem 3.1 is now established and so is the main theorem.

We are also in a position now to show that the M' property characterises ellipses.

THEOREM 5.2. *Suppose that Γ is a smooth, strictly convex curve which has property M' about some point in its interior, then Γ is an ellipse, and the point is the point of symmetry.*

PROOF. By translation, we may assume that Γ has the M' property about $\mathbf{0}$. Let $p(t) = \rho(t)e^{it}$ be the angle parametrisation about $\mathbf{0}$.

We define a function N on $\mathbf{R}^2 - \{\mathbf{0}\}$ by

$$N(re^{i\theta}) = r/\rho(\theta).$$

Since Γ is convex, $N(\boldsymbol{x} + \boldsymbol{y}) \leqq N(\boldsymbol{x}) + N(\boldsymbol{y})$ and clearly $N(r\boldsymbol{x}) = rN(\boldsymbol{x})$ for $r > 0$.

In case Γ is symmetric about $\mathbf{0}$, we have that N is a smooth, strictly convex norm whose unit sphere is Γ. By Lemma 3.1, Γ is rigid, and by Theorem 3.1, N is a linear translate of the Euclidean norm, whence Γ is an ellipse with $\mathbf{0}$ its point of symmetry.

To see that Γ is necessarily symmetric about $\mathbf{0}$, note first that $\gamma\theta(s, t) = t$ if and only if $t = s + \pi$. Next,

$$\begin{aligned} \gamma\theta(\gamma^{-1}(s), \gamma^{-1}(s + \pi)) &= \theta(s, s + \pi) && \text{by the } M' \text{ property} \\ &= \gamma^{-1}(s + \pi) && \text{by the above.} \end{aligned}$$

Thus, $\gamma^{-1}(s + \pi) = \gamma^{-1}(s) + \pi$, and Γ is symmetric about $\mathbf{0}$. □

§6. The problem in higher dimension

The general problem is to characterize rigid sets in finite dimensional Banach spaces. There are infinite rigid sets in $\mathbf{R}^n$ ($n \geqq 3$) with respect to non-Hilbertian norms (private communications of P. Milman and R. Wittmann). We consider here the situation where $\| \ \|$ is a smooth, strictly convex norm on $\mathbf{R}^n$ whose unit sphere is rigid.

THEOREM 6.1. *If $\| \ \|$ is a strictly convex, smooth norm on $\mathbf{R}^n$, whose unit sphere is rigid, and n is odd, then $\| \ \|$ is Hilbertian.*

Theorem 6.1 follows easily from a result of M. Gromov (theorem 1(1) [3]) by way of Lemma 6.2 (below). Gromov's result (theorem 1 in [3]) is a partial proof of the conjecture of Banach that if $2 \leqq k < n$, and all k-dimensional subspaces of an n-dimensional Banach space are isometric, then the Banach space is a Hilbert space. (See also [2], p. 152.)

LEMMA 6.2. *If $n \geqq 3$, $\| \ \|$ is a smooth, strictly convex norm on $\mathbf{R}^n$ whose unit sphere is rigid, then all $(n-1)$-dimensional subspaces of $(\mathbf{R}^n, \| \ \|)$ are isometric.*

PROOF. If the isometries of the unit sphere of $\| \ \|$ are differentiable mappings, then their differentials are linear mappings between appropriate tangent spaces, which are easily seen to be isometries when the tangent spaces are equipped with the norms inherited from $(\mathbf{R}^n, \| \ \|)$. By strict convexity (and smoothness) every $(n-1)$-dimensional subspace appears as a tangent space, and any two are isometric by the transitivity of the isometries. We complete this proof by showing that an isometry g of Γ (the unit sphere of $\| \cdot \| = N(\cdot)$) is indeed smooth.

The Taylor expansion of the formula

$$N(g(x') - g(y)) - N(g(x) - g(y)) = N(x' - y) - N(x - y)$$

for $x, x', y \in \Gamma$, $x \neq y$ and x' close to x, yields

$$\nabla N(g(x) - g(y)) \cdot (g(x') - g(x)) = \nabla N(x - y) \cdot (x' - x) + o(\| x' - x \|).$$

Dividing by $\| x' - x \|$ and letting $x' \to x$ so that $x' - x / \| x' - x \| \to v \in T_x\Gamma$, we obtain that

$$\nabla N(g(x) - g(y)) \cdot \left(\frac{g(x') - g(x)}{\| x' - x \|} \right) \to \nabla N(x - y) \cdot v.$$

As y passes through $\Gamma - \{x\}$, $\nabla N(g(x) - g(y))$ passes through a spanning set for $\mathbf{R}^n$, whence

$$\frac{g(x') - g(x)}{\| x' - x \|} \to L(v) \in T_{g(x)}\Gamma$$

where $\nabla N(g(x) - g(y)) \cdot L(v) = \nabla N(x - y) \cdot v$. Clearly, $L(v)$ can be so constructed for any $v \in T_x\Gamma$, $\| v \| = 1$. The linearity of L as a function of V and its continuity as a function of x follow for this latter equation, whence g is smooth with derivative L. □

Thus, the truth of Banach's conjecture would imply that the only smooth, strictly convex norms on $\mathbf{R}^n$ ($n \geqq 3$) with rigid unit sphere are Hilbertian.

M. Gromov proved the conjecture of Banach for k even (theorem 1(1) [3]), whence our Theorem 6.1.

References

1. M. Akcoglu and U. Krengel, *Nonlinear models of diffusion on a finite space*, Probab. Relat. Fields **76** (1987), 411–420.

2. D. Amir, *Characterization of Inner Product Spaces*, Birkhäuser, Basel, Boston, Stuttgart, 1986.

3. M. Gromov, *A geometrical conjecture of Banach*, Math. USSR Izv. **1** (1967), 1055–1064.

4. R. Sine, *A non-linear Perron–Frobenius theorem*, Proc. Am. Math. Soc., to appear.